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FAST TRACK COMMUNICATION

T-duality as a duality of loop group bundles

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Abstract
Representing the data of a string compactified on a circle in the background of H-flux in terms of the geometric data of a principal loop group bundle, we show that T-duality in type II string theory can be understood as the interchange of the momentum and winding homomorphisms of the principal loop group bundle, thus giving rise to a new interpretation of T-duality.

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1. Introduction

T-duality is a striking property of type II string theory on compactified spacetimes. In earlier papers [4], we gave a prescription for the topological aspects of T-duality in type II string theory for principal circle bundles with H-flux, and later on [5, 13], for higher rank principal torus bundles. In this communication, we will mainly concentrate on the case of principal circle bundles. In the physics literature, this is the same as saying that spacetime is compactified in one direction.

We begin by reviewing our results in the case of circle bundles [4, 15]. Let

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & Y \\ & & \Big\downarrow{\scriptstyle \pi} \\ & & X \end{array} \qquad (1.1)$$

be a principal circle bundle and H a closed 3-form on Y with integral periods. The T-dual of (Y, H) is another spacetime $\widehat{Y}$ with H-flux $\widehat{H}$, with the following desirable properties:

(i) the RR-fields and charges are mapped in a one-to-one manner respectively;
(ii) T-duality applied twice returns one to the original spacetime Y with original H-flux.

In our earlier papers, *op. cit.*, we generalized the Buscher rules, which are the local transformation rules of the low-energy effective fields under T-duality as given in e.g. [8, 12], to the case when H defines a topologically non-trivial cohomology class. The result can be described as follows: $\int_{\mathbb{T}} H$ is a closed 2-form on X with integral periods, and therefore it is a representative of the first Chern class $c_1(\widehat{Y})$ of a principal circle bundle

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & \widehat{Y} \\ & & \big\downarrow \widehat{\pi} \\ & & X \end{array} \tag{1.2}$$

In fact, geometric prequantization asserts that $\int_{\mathbb{T}} H$ is the curvature of a connection on $\widehat{Y}$. The T-dual flux $\widehat{H}$ should have the property that the closed 2-form $\int_{\widehat{\mathbb{T}}} \widehat{H}$ with integral periods represents the first Chern class $c_1(Y)$ of the principal circle bundle Y. However, this does not characterize $\widehat{H}$, since the lift of any closed 3-class on X to $\widehat{Y}$ integrates to zero. To determine $\widehat{H}$, we need an additional condition, which is stated on the correspondence space $Y \times_X \widehat{Y}$, defined by the commutative diagram,

$$\begin{array}{ccccc} & & Y \times_X \widehat{Y} & & \\ & \swarrow^{p} & & \searrow^{\widehat{p}} & \\ Y & & & & \widehat{Y} \\ & \searrow_{\pi} & & \swarrow_{\widehat{\pi}} & \\ & & X & & \end{array} \tag{1.3}$$

The additional condition that determines the cohomology class of the T-dual flux $\widehat{H}$ is that $p^*[H] = \widehat{p}^*[\widehat{H}] \in H^3(Y \times_X \widehat{Y}, \mathbb{Z})$. The Gysin sequence for the circle bundle (1.2) establishes the existence of the T-dual flux.

One of the astonishing consequences of our theorem is that in general, there is a *change in topology* between T-dual spacetimes with T-dual background flux. Moreover, T-duality can be interpreted as the duality *exchanging the H-flux with the Chern class*.

Using the fact that $H^3(Y, \mathbb{Z})$ classifies (isomorphism classes) of principal $\mathsf{PU}(\mathcal{H})$-bundles over Y, we can encode the data representing a circle bundle with background flux geometrically as a principal $\mathsf{PU}(\mathcal{H})$-bundle $\tilde{\pi} : P \to Y$ over a principal $\mathbb{T}$-bundle $\pi : Y \to X$. One main goal of this communication is to give a direct prescription of the T-dual spacetime and flux using this geometric set-up.

The paper is organized as follows. In section 2, we reformulate the geometric data in terms of certain principal loop group bundles along the lines of [14, 7]. Then, in section 3, we interpret T-duality for principal circle bundles in a background H-flux, as a duality of principal loop group bundles *exchanging the momentum and winding homomorphisms*, and make contact to the classifying space analysis of [7] in section 4. Our results are summarized in theorem 4.1.

The results of this paper are in essence an amalgamation of the results of, in particular, [2, 7], and based on ideas in [10, 4]. We feel, however, that it is useful to emphasize this geometric interpretation of T-duality as it has various other potential applications, and in principle allows for the use of differential geometric techniques familiar to most physicists (see, e.g., [14, 16] for the computation of characteristic classes of loop group bundles). Other geometrizations of the T-duality data are possible, and have been studied in, for example, [1].

2. Reformulation as loop group bundles

In this section, we will review the 1–1 correspondence between (isomorphism classes of) principal G-bundles P over principal $\mathbb{T}$-bundles Y over X, and (isomorphism classes of) principal $L\mathsf{G} \rtimes \mathbb{T}$-bundles over X, under the assumption that G is simply connected (i.e. $\pi_1(\mathsf{G}) = 0$) [14, 2].

$$\left(\begin{array}{ccc} \mathsf{G} & \longrightarrow & P \\ & & \downarrow \tilde{\pi} \\ \mathbb{T} & \longrightarrow & Y \\ & & \downarrow \pi \\ & & X \end{array}\right) \iff \left(\begin{array}{ccc} L\mathsf{G} \rtimes \mathbb{T} & \longrightarrow & Q \\ & & \downarrow \Pi \\ & & X \end{array}\right) \tag{2.1}$$

We recall the definition of $L\mathsf{G} \rtimes \mathbb{T}$. First of all, $\mathbb{T}$ acts on $L\mathsf{G}$ by $\mathbb{T} \times L\mathsf{G} \to L\mathsf{G}$, $(t, \gamma) \mapsto t \cdot \gamma$, where $(t \cdot \gamma)(s) = \gamma(ts)$. The semi-direct product $L\mathsf{G} \rtimes \mathbb{T}$ is then defined by the multiplication law $(\gamma_1, t_1) \circ (\gamma_2, t_2) = ((t_2 \cdot \gamma_1)\gamma_2, t_1 t_2)$. Equivalently, we can think of the semi-direct product $L\mathsf{G} \rtimes \mathbb{T}$ as the split short exact sequence

$$1 \longrightarrow L\mathsf{G} \xrightarrow{\imath} L\mathsf{G} \rtimes \mathbb{T} \underset{s}{\overset{\rho}{\rightleftarrows}} \mathbb{T} \longrightarrow 1 \tag{2.2}$$

where, for $(\gamma, t) \in L\mathsf{G} \rtimes \mathbb{T}$ we have $\rho(\gamma, t) = t$, and $\imath(\gamma) = (\gamma, 1)$. We will refer to ρ as the 'momentum homomorphism'.

We will first discuss the correspondence (2.1) explicitly, using transition functions. Let $\{U_i\}$ be a good cover of X for which we have trivializations $\phi_i : \pi^{-1}(U_i) \xrightarrow{\simeq} U_i \times \mathbb{T}$. We write $\phi_i(y) = (\pi(y), s_i(y))$, where the 'section' $s_i : \pi^{-1}(U_i) \to \mathbb{T}$ satisfies $s_i(yt) = s_i(y)t$, for $t \in \mathbb{T}$ (group action written multiplicatively). The transition functions $g_{ij} : U_{ij} \to \mathbb{T}$ are defined by

$$\left(\phi_i \circ \phi_j^{-1}\right)(x, t) = (x, g_{ij}(x)t), \quad \text{i.e.} \quad g_{ij}(x) = s_i(y)s_j(y)^{-1}, \tag{2.3}$$

where $y \in \pi^{-1}(x) \subset Y$, and we write multiple intersections as $U_{ij} = U_i \cap U_j$, etc. This definition does not depend on the choice of $y \in \pi^{-1}(x)$. The transition functions satisfy the cocycle identity, $g_{ij}(x)g_{jk}(x) = g_{ik}(x), x \in U_{ijk}$. We also recall that the $\mathbb{T}$-bundle can be reconstructed from the transition functions by setting $E = \coprod_i (U_i \times \mathbb{T})/\sim$, where we identify $(x, t) \sim (x, t')$ on $U_{ij} \times \mathbb{T}$ iff $t = g_{ij}(x)t'$.

Let $V_i = \pi^{-1}(U_i)$. Since V_i is homotopic to $\mathbb{T}$, and since $\pi_1(\mathsf{G}) = 0$ by assumption, the G-bundle $\tilde{\pi} : P \to Y$ trivializes over V_i. Denote the local trivialization by $\tilde{\phi}_i : \tilde{\pi}^{-1}(V_i) \xrightarrow{\simeq} V_i \times \mathsf{G}$, the corresponding section by $\tilde{s}_i : \tilde{\pi}^{-1}(V_i) \to \mathsf{G}$ and the transition functions by $\tilde{g}_{ij} : V_{ij} \to \mathsf{G}$.

We will now show how to use these data to define a principal $L\mathsf{G} \rtimes \mathbb{T}$-bundle Q over M. We define it by declaring that the transition functions $G_{ij} : U_{ij} \to L\mathsf{G} \times \mathbb{T}$, are given by

$$G_{ij}(x) = \left(\tilde{g}_{ij}\left(\phi_j^{-1}(x, \cdot)\right), g_{ij}(x)\right), \qquad x \in U_{ij}. \tag{2.4}$$

Then, for $x \in U_{ijk}$, one has

$$\begin{aligned} G_{ij}(x)G_{jk}(x) &= \left(\tilde{g}_{ij}\left(\phi_j^{-1}(x, \cdot)\right), g_{ij}(x)\right) \circ \left(\tilde{g}_{jk}\left(\phi_k^{-1}(x, \cdot)\right), g_{jk}(x)\right) \\ &= \left(g_{jk}(x) \cdot \tilde{g}_{ij}\left(\phi_j^{-1}(x, \cdot)\right)\tilde{g}_{jk}\left(\phi_k^{-1}(x, \cdot)\right), g_{ij}(x)g_{jk}(x)\right) \end{aligned}$$

$$= \big(\tilde g_{ij}\big(\phi_k^{-1}(x,\cdot)\big)\tilde g_{jk}\big(\phi_k^{-1}(x,\cdot)\big), g_{ij}(x)g_{jk}(x)\big)$$
$$= \big(\tilde g_{ik}\big(\phi_k^{-1}(x,\cdot)\big), g_{ik}(x)\big) = G_{ik}(x), \tag{2.5}$$

where we have used the cocycle properties of the transition functions g_{ij} and $\tilde g_{ij}$. There is an intermediate step in this construction, namely that of a $\mathbb{T}$-equivariant principal $L\mathsf{G}$-bundle LP over Y. This bundle is defined by its transition functions

$$\tilde G_{ij}(y)(t) = \tilde g_{ij}(yt), \qquad y \in V_{ij}, \quad t \in \mathbb{T}. \tag{2.6}$$

Note that the transition functions satisfy

$$\tilde G_{ij}(ys)(t) = \tilde g_{ij}((ys)t) = \tilde g_{ij}(y(st)) = \tilde G_{ij}(y)(st)$$
$$= (s \cdot G_{ij}(y))(t), \tag{2.7}$$

proving their equivariance. Conversely, given a $\mathbb{T}$-equivariant principal $L\mathsf{G}$-bundle LP over Y, we define a principal G-bundle by the transition functions $\tilde g_{ij}(y) = \tilde G_{ij}(y)(1)$, i.e. by evaluating the loop in $L\mathsf{G}$ at $t = 1$.

2.1. Reconstruction

Conversely, given a principal $L\mathsf{G} \rtimes \mathbb{T}$-bundle Q over X with transition functions $G_{ij} : U_{ij} \to L\mathsf{G} \rtimes \mathbb{T}$, we can reconstruct the transition functions of a $\mathbb{T}$-bundle Y over X and the G-bundle P over Y as follows. First we let $g_{ij} = \rho(G_{ij})$ (cf (2.2)) be the transition function of the principal $\mathbb{T}$-bundle $\pi : Y \to X$, and $\tilde g_{ij}(y) = \jmath(G_{ij}(\pi(y)))(s_j(y))$ the transition function of the G-bundle, where $\jmath(\gamma, t) = \gamma$ is a left splitting of (2.2). This construction clearly is the inverse of the construction described above. There is a more concrete way of describing the reconstruction which avoids the explicit use of transition functions (see, e.g., [14]). Let Q be an $L\mathsf{G} \rtimes \mathbb{T}$ principal bundle. We define a free action of $L\mathsf{G} \rtimes \mathbb{T}$ on $Q \times \mathsf{G} \times \mathbb{T}$ by $(\gamma, t) \cdot (\tilde p, g, s) = (R_{(\gamma,t)}\tilde p, \gamma(s)g, st)$. That this defines an action of $L\mathsf{G} \rtimes \mathbb{T}$ can be seen as follows:

$$\begin{aligned}(\gamma_1, t_1) \cdot ((\gamma_2, t_2) \cdot (\tilde p, g, s)) &= (\gamma_1, t_1) \cdot \big(R_{(\gamma_2,t_2)}\tilde p, \gamma_2(s)g, st_2\big)\\ &= \big(R_{(\gamma_1,t_1)}R_{(\gamma_2,t_2)}\tilde p, \gamma_1(st_2)\gamma_2(s)g, st_2), st_1t_2\big)\\ &= \big(R_{(\gamma_1,t_1)\circ(\gamma_2,t_2)}\tilde p, ((\gamma_1, t_1) \circ (\gamma_2, t_2))(s), st_1t_2\big)\\ &= ((\gamma_1, t_1) \circ (\gamma_2, t_2)) \cdot (\tilde p, g, s).\end{aligned}$$

We denote by $[\tilde p, g, s]$ the equivalence class of triples under the action of $L\mathsf{G} \rtimes \mathbb{T}$. It is easily seen that $Q \times \mathsf{G} \times \mathbb{T}$ also carries a free action of G, commuting with the $L\mathsf{G} \rtimes \mathbb{T}$ action, namely

$$R_h(\tilde p, g, s) = (\tilde p, gh^{-1}, s),$$

and hence a free G-action on $(Q \times \mathsf{G} \times \mathbb{T})/L\mathsf{G} \rtimes \mathbb{T}$. This makes $(Q \times \mathsf{G} \times \mathbb{T})/L\mathsf{G} \rtimes \mathbb{T}$ into a principal G-bundle over $(Q \times \mathbb{T})/L\mathsf{G} \rtimes \mathbb{T}$. Similarly, we conclude that $(Q \times \mathbb{T})/L\mathsf{G} \rtimes \mathbb{T}$ is a principal $\mathbb{T}$-bundle over $Q/L\mathsf{G} \rtimes \mathbb{T} = X$. The situation is summarized below

$$\begin{array}{ccccc}\mathsf{G} & \longrightarrow & (Q \times \mathsf{G} \times \mathbb{T})/L\mathsf{G} \rtimes \mathbb{T} & & [\tilde p, g, s]\\ & & \downarrow{\scriptstyle\tilde\pi} & & \downarrow\\ \mathbb{T} & \longrightarrow & (Q \times \mathbb{T})/L\mathsf{G} \rtimes \mathbb{T} & & [\tilde p, s]\\ & & \downarrow{\scriptstyle\pi} & & \downarrow\\ & & Q/L\mathsf{G} \rtimes \mathbb{T} & & [\tilde p]\end{array}$$

3. T-duality of loop group bundles

In this section, we will consider specifically the projective unitary group $\mathsf{G} = \mathsf{PU}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$, which is the geometric set-up for the discussion of T-duality for $\mathbb{T}$-bundles with H-flux, as discussed in the introduction. In this case, we will see that there is an additional circle 'hidden' in the $L\mathsf{G} \rtimes \mathbb{T}$-bundle description and that by interchanging the role of the circles (interchanging momentum and winding) before applying the reconstruction process produces the T-dual bundles. Schematically

$$\left(\begin{array}{ccc} \mathsf{G} & \longrightarrow & P \\ & & \big\downarrow \tilde{\pi} \\ \mathbb{T} & \longrightarrow & Y \\ & & \big\downarrow \pi \\ & & X \end{array}\right) \Rightarrow \left(\begin{array}{ccc} L\mathsf{G} \rtimes \mathbb{T} & \longrightarrow & Q \\ & & \big\downarrow \Pi \\ & & X \end{array}\right) \Rightarrow \left(\begin{array}{ccc} \mathsf{G} & \longrightarrow & \widehat{P} \\ & & \big\downarrow \widehat{\tilde{\pi}} \\ \mathbb{T} & \longrightarrow & \widehat{Y} \\ & & \big\downarrow \widehat{\pi} \\ & & X \end{array}\right)$$

First recall that the loop group $L\mathsf{G} = \mathrm{Map}(\mathbb{T}, \mathsf{G})$, and the based loop group $\Omega\mathsf{G} = \{\gamma \in L\mathsf{G} | \gamma(0) = e\}$, where $e \in \mathsf{G}$ is the identity element, are related by the isomorphism of groups $L\mathsf{G} \cong \Omega\mathsf{G} \rtimes \mathsf{G}$, $\gamma(\cdot) \mapsto (\gamma(\cdot)\gamma(0)^{-1}, \gamma(0))$, where the action of G on $\Omega\mathsf{G}$ is given by $(g \cdot \gamma)(\cdot) = g\gamma(\cdot)g^{-1}$. Now observe that

$$\begin{aligned} \pi_1(L\mathsf{G} \rtimes \mathbb{T}) &\cong \pi_1(L\mathsf{G}) \oplus \pi_1(\mathbb{T}) \cong \pi_1(\Omega\mathsf{G} \rtimes \mathsf{G}) \oplus \pi_1(\mathbb{T}) \\ &\cong \pi_1(\mathsf{G}) \oplus \pi_1(\Omega\mathsf{G}) \oplus \pi_1(\mathbb{T}) \cong \pi_1(\mathsf{G}) \oplus \pi_2(\mathsf{G}) \oplus \pi_1(\mathbb{T}). \end{aligned}$$

Now, if G is a simply connected (as we have assumed previously), compact Lie group, then $\pi_1(\mathsf{G}) \cong \pi_2(\mathsf{G}) \cong 0$ and hence $\pi_1(L\mathsf{G} \rtimes \mathbb{T}) \cong \mathbb{Z}$, i.e. the group $L\mathsf{G} \rtimes \mathbb{T}$ 'contains' only one circle described by equation (2.2). If, however, $\mathsf{G} = \mathsf{PU}(\mathcal{H})$, then $\pi_2(\mathsf{G}) \cong \mathbb{Z}$ and thus $\pi_1(L\mathsf{G} \rtimes \mathbb{T}) \cong \mathbb{Z} \oplus \mathbb{Z}$ which means we have two circles sitting inside $L\mathsf{G} \rtimes \mathbb{T}$.

The second circle can be recovered as follows. The group $\mathsf{G} = \mathsf{PU}(\mathcal{H})$ has a canonical central extension $\mathsf{U}(\mathcal{H})$

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathsf{U}(\mathcal{H}) \longrightarrow \mathsf{PU}(\mathcal{H}) \longrightarrow 1 \tag{3.1}$$

The central extension (3.1) has a connection compatible with the group structure. The holonomy of this connection is a homomorphism $\mathrm{hol}\colon L\mathsf{G} \to \mathbb{T}$. We let $\mathsf{N} = \mathrm{Ker}(\mathrm{hol} : L\mathsf{G} \to \mathbb{T})$, a normal subgroup of $L\mathsf{G}$. We thus have an exact sequence

$$1 \longrightarrow \mathsf{N} \rtimes \mathbb{T} \xrightarrow{\iota} L\mathsf{G} \rtimes \mathbb{T} \xrightarrow{\omega} \mathbb{T} \longrightarrow 1 \tag{3.2}$$

where ω, the winding homomorphism, is defined by $\omega(\gamma, t) = \mathrm{hol}(\gamma)$. If this were again a split exact sequence, like (2.2), we could apply the reconstruction theorem and obtain a dual G-bundle over a dual $\mathbb{T}$-bundle. However, the sequence (3.2) is not split and hence $\mathsf{N} \rtimes \mathbb{T}$ is not isomorphic to $L\mathsf{G}$. From the exact sequence in homotopy it follows easily, however, that $\mathsf{N} \rtimes \mathbb{T}$ and $L\mathsf{G}$ are homotopy equivalent. After performing this homotopy, we can perform the reconstruction, the result being our T-dual circle bundle with T-dual H-flux described in the introduction. A proof of this statement follows from the work of Bunke and Schick [7]. The connection of the above discussion to this work is the subject of the following section.

4. Classifying maps

Let us reformulate the equivalence (2.1) in terms of classifying maps, specialize to the case $\mathsf{G} = \mathsf{PU}(\mathcal{H})$ and make the connection to the work of [7].

The principal $\mathbb{T}$-bundle $\pi : Y \to X$ gives us a map $\widetilde{\phi} : X \to B\mathbb{T}$, while the principal G-bundle $\tilde{\pi} : P \to Y$ gives us a map $\psi : Y \to B\mathsf{G}$. We have already seen that the principal G-bundle $\tilde{\pi} : P \to Y$ is equivalent to a $\mathbb{T}$-equivariant principal $L\mathsf{G}$ bundle $\widehat{\pi} : LP \to Y$, hence a classifying map $\widehat{\psi} : Y \to BL\mathsf{G}$ satisfying $\widehat{\psi}(yt) = t \cdot \widehat{\psi}(y)$. Also, we can construct a map $\widehat{\phi} : Y \to E\mathbb{T}$, covering $\widetilde{\phi}$. To summarize

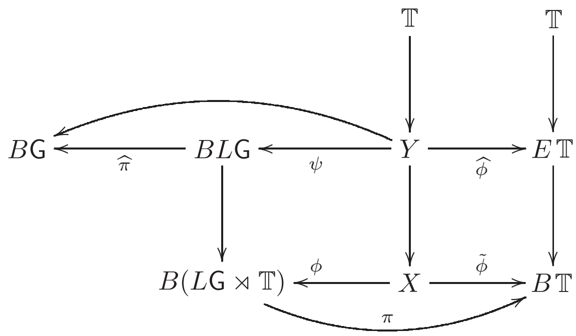
$$\begin{array}{ccc} Y & \xrightarrow{\widehat{\phi}} & E\mathbb{T} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{\tilde{\phi}} & B\mathbb{T} \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\widehat{\psi}} & BL\mathsf{G} \\ {\scriptstyle\psi}\downarrow & & \\ B\mathsf{G} & & \end{array}$$

The two maps $\widehat{\phi}$ and $\widehat{\psi}$ can be combined into a map $\Phi = (\widehat{\phi}, \widehat{\psi}) : Y \to E\mathbb{T} \times BL\mathsf{G}$, which descends to a map $\phi : X \to E\mathbb{T} \times_{\mathbb{T}} BL\mathsf{G}$, since $\phi(yt) = [\widehat{\phi}(yt), \widehat{\psi}(yt)] = [\widehat{\phi}(y)t, t\cdot\widehat{\psi}(y)] = [\widehat{\phi}(y), \widehat{\psi}(y)] = \phi(y)$, where [,] denotes equivalence classes of pairs in $E\mathbb{T} \times BL\mathsf{G}$ under the action of $\mathbb{T}$. Let us denote $R = E\mathbb{T} \times_{\mathbb{T}} BL\mathsf{G}$.

Conversely, suppose we are given a map $\phi : X \to R$. Since $\mathbb{T}$ acts freely on $E\mathbb{T}$ we have a projection $\widehat{\pi} : R \to B\mathbb{T}$, i.e. R is a principal $BL\mathsf{G}$ bundle over $B\mathbb{T}$.

$$\begin{array}{ccc} BL\mathsf{G} & \longrightarrow & R \\ & & \downarrow{\scriptstyle\widehat{\pi}} \\ & & B\mathbb{T} \end{array}$$

Thus we have a map $\tilde{\phi} = \widehat{\pi} \circ \phi : X \to B\mathbb{T}$. This map defines our principal $\mathbb{T}$-bundle $\pi : Y \to X$. We can lift $\widetilde{\phi}$ to $\widehat{\phi} : Y \to E\mathbb{T}$, and then define a map $\widehat{\psi} : Y \to BL\mathsf{G}$ uniquely by $\phi(\pi(y)) = [\widehat{\phi}(y), \widehat{\psi}(y)]$, and we conclude that $\widehat{\psi}$ is $\mathbb{T}$-equivariant. Finally, we obtain a map $\psi : Y \to B\mathsf{G}$, defining our principal G-bundle over Y, by using that $BL\mathsf{G} \cong B(\Omega\mathsf{G} \rtimes \mathsf{G}) \cong E\mathsf{G} \times_{\mathsf{G}} B\Omega\mathsf{G}$, so that in particular we have a map $\widehat{\pi} : BL\mathsf{G} \to B\mathsf{G}$. To summarize, we have shown that $R \cong B(L\mathsf{G} \rtimes \mathbb{T})$, i.e. the map ϕ classifies principal $L\mathsf{G} \rtimes \mathbb{T}$-bundles over X. The various maps are summarized in the diagram below



Now we specialize to $\mathsf{G} = \mathsf{PU}(\mathcal{H})$. We recall that $\mathbb{T} = K(\mathbb{Z}, 1)$ and $\mathsf{G} = K(\mathbb{Z}, 2)$, hence $B\mathbb{T} = K(\mathbb{Z}, 2)$ and $B\mathsf{G} = K(\mathbb{Z}, 3)$. We have seen that $B(L\mathsf{G} \rtimes \mathbb{T})$, the classifying space for principal $L\mathsf{G} \rtimes \mathbb{T}$-bundles over X is isomorphic to $R = E\mathbb{T} \times_{\mathbb{T}} BL\mathsf{G}$, and that R therefore has the natural structure of a principal $BL\mathsf{G} \cong (K(\mathbb{Z}, 3) \rtimes K(\mathbb{Z}, 2))$-bundle over $B\mathbb{T} = K(\mathbb{Z}, 2)$. However, there is another way of interpreting R, namely as a $K(\mathbb{Z}, 3)$ homotopy fibration over $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2)$ [7] (see also [13]). Moreover, there is a map $T : R \to R$ such that $T^* : H^2(R, \mathbb{Z}) \to H^2(R, \mathbb{Z})$ exchanges the two generators, and $T \circ T$ is homotopic to the identity on R. It turns out that $T : R \to R$ implements T-duality for principal circle bundles with background flux (see [7]): a principal $L\mathsf{G} \rtimes \mathbb{T}$-bundle Q over X has two natural characteristic classes of degree 2 on X. One of these is the first Chern class of the associated circle bundle over X, $c_1(Y)$, and the other is given by integration over the fiber of Y, of the Dixmier–Douady invariant of P. We denote these by $c(Q)$ and $d(Q)$ respectively, and they are the pullback under the classifying map $\phi : X \to R$ of the generators of $H^2(R, \mathbb{Z})$.

Hence, in terms of classifying maps, the T-dual principal $L\mathsf{G} \rtimes \mathbb{T}$-bundle $\widehat{Q}$ over X is defined by considering the continuous map $T \circ f : X \to B(L\mathsf{G} \rtimes \mathbb{T})$, and by associating with it $\widehat{Q} = (T \circ f)^*(E(L\mathsf{G} \rtimes \mathbb{T}))$. It follows that T-duality exchanges the entries of the pair $(c(Q), d(Q))$, and that T-duality applied twice gives a bundle that is isomorphic to Q, since $T \circ T \sim \mathrm{I}_R$. We summarize this as follows.

Theorem 4.1 (T-duality as a duality of principal loop group bundles)**.** *Given a principal loop group bundle,*

$$\begin{array}{ccc} L\mathsf{G} \rtimes \mathbb{T} & \longrightarrow & Q \\ & & \big\downarrow \Pi \\ & & X \end{array}$$

with classifying map $\phi : X \to B(L\mathsf{G} \rtimes \mathbb{T})$, then there exists a T-dual loop group principal bundle,

$$\begin{array}{ccc} L\mathsf{G} \rtimes \mathbb{T} & \longrightarrow & \widehat{Q} \\ & & \big\downarrow \widehat{\Pi} \\ & & X \end{array}$$

with classifying map $T \circ \phi : X \to B(L\mathsf{G} \rtimes \mathbb{T})$, which has the following properties:

(i) $\widehat{\widehat{Q}}$ is isomorphic to Q;
(ii) $c(\widehat{Q}) = d(Q)$ and $d(\widehat{Q}) = c(Q)$.

Geometrically speaking, T-duality can be viewed as the exchange of the momentum and winding homomorphisms of the previous section (see equations (2.2) and (3.2)).

5. Concluding remarks

The RR-fields in type IIA string theory can also be interpreted as principal loop group bundles over X. To this end, recall that in theorem 7.2(5) of [3], the following description of twisted K-theory is given. Every element of twisted K-theory, twisted by a principal G-bundle P over Y (here as before, $\mathsf{G} = \mathsf{PU}(\mathcal{H})$), is given by a principal $\mathsf{U}_{\mathcal{K}} \rtimes \mathsf{G}$ bundle A over Y that projects onto P. Here $\mathsf{U}_{\mathcal{K}}$ denotes the group of unitary operators in the given Hilbert space which are of the form identity plus compact operator (known as the universal gauge group [11]), and the projection condition means that $P \cong A \times_{\mathsf{U}_{\mathcal{K}} \rtimes \mathsf{G}} \mathsf{G}$. Therefore RR-fields can be interpreted as principal $L(U_{\mathcal{K}} \rtimes \mathsf{G}) \rtimes \mathbb{T}$-bundles over X that project onto the given principal $L(\mathsf{G}) \rtimes \mathbb{T}$-bundle Q over X. Similarly, charges of type IIA RR-fields can be interpreted as principal $\Omega \mathsf{U}_{\mathcal{K}} \rtimes \mathsf{G}$ bundles B over Y that project onto Q. We conclude that the T-duality isomorphism of the RR-fields and their charges in type II string theory as established in [4], can be also interpreted as a duality isomorphism between loop group bundles on X.

Finally, the analysis of this paper can be generalized to T-duality for higher rank torus bundles, provided the principal G-bundle P is trivial when restricted to the torus fiber of $Y \to X$. This requires the H-flux to be classical, in the terminology of [5], and replaces the condition that $\pi_1(\mathsf{G}) = 0$.

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